6.1 Inner Product, Length, and Orthogonality.

Consider the two set of Vectors in $\mathbb{R}^3$

1) $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$.

2) $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$

Compute

$$\vec{v}_1^T \vec{v}_2,\ \vec{v}_2^T \vec{v}_2,\ \vec{u}_2^T \vec{u}_3, \text{ and } \vec{u}_3^T \vec{u}_2$$

Answ:

$$\vec{v}_1^T \vec{v}_2 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2 - 2 = 0, \quad \vec{v}_2^T \vec{v}_2 = \begin{bmatrix} 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = $$
$$= 2^2 + 0^2 + 2^2 = 4 + 4 = 8$$

$$\vec{u}_2^T u_3 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = 4, \quad \vec{u}_3^T \vec{u}_2 = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 4$$

Def.- $\vec{u}, \vec{v} \in \mathbb{R}^n$

The real number $\vec{u}^T \vec{v}$ is called the Inner Product of $\vec{u}$ and $\vec{v}$. It is also written as $\vec{u} \cdot \vec{v}$.

It means

$$\boxed{\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}}$$

It is also called "the dot product".

Notice that

$$\vec{u}_2 \cdot \vec{u}_3 = \vec{u}_3 \cdot \vec{u}_2 .$$

This is true in general.

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

then $\vec{v} \cdot \vec{v} = \vec{v}^T \vec{v} =$

$$= [v_1\ v_2\ v_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1^2 + v_2^2 + v_3^2$$

<u>OTHER PROPERTIES:</u>

<u>Thm 1</u> $\vec{u}, \vec{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$

a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$.

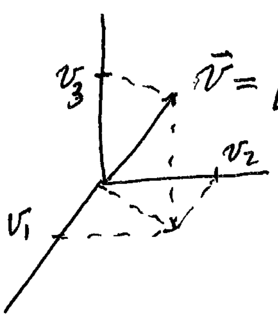
d) $\vec{u} \cdot \vec{u} \geq 0$, and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = 0$.

<u>Proof.</u> All based in properties of real numbers.

<u>Length of a Vector</u>

<u>Notation</u>: Lenght of $\vec{v}$: $\|\vec{v}\|$

From previous knowledge in $\mathbb{R}^3$, $\mathbb{R}^2$



$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\vec{v} \cdot \vec{v}}$$

For example,

$$\|\vec{v}_2\| = \left\| \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\| = \sqrt{2^2 + 0^2 + 2^2} = \sqrt{8}$$

Also notice, $\vec{v}_2 \cdot \vec{v}_2 = \vec{v}_2^T \vec{v}_2 = [2\ 0\ 2] \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 4 + 4 = 8$

$$\Rightarrow \|\vec{v}_2\| = \sqrt{\vec{v}_2 \cdot \vec{v}_2} .$$

Def. For $v \in \mathbb{R}^n$ $\quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

We define
Length of vector $\vec{v}$ as $\boxed{\|\vec{v}\| = \sqrt{(\vec{v}\cdot\vec{v})} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}}$

Notice, $\quad \|\vec{v}\|^2 = v_1^2 + \cdots + v_n^2 = \vec{v}\cdot\vec{v}$

For any $c \in \mathbb{R}$: $\quad \|c\vec{v}\| = |c|\,\|\vec{v}\| \qquad (3.1)$

Unit vectors : Any vector $\vec{v}$ such that $\|\vec{v}\| = 1$.

$\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ is not a unit vector since

$$\|\vec{v}_2\| = \sqrt{2^2 + 0^2 + 2^2} = \sqrt{8}$$

but, $\quad \vec{w} = \dfrac{\vec{v}_2}{\|\vec{v}_2\|}$ is a unit vector since from (3.1)

$$\|\vec{w}\| = \frac{1}{\|\vec{v}_2\|}\,\|\vec{v}_2\| = 1. \qquad = \begin{bmatrix} 2/\sqrt{8} \\ 0 \\ 2/\sqrt{8} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

Def.- Given a vector $\vec{v} \neq \vec{0} \in \mathbb{R}^n$ such that $\|\vec{v}\| \neq 1$
A unit vector with "the same direction" as $\vec{v}$ is given by

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}.$$

DISTANCE

From previous knowledge

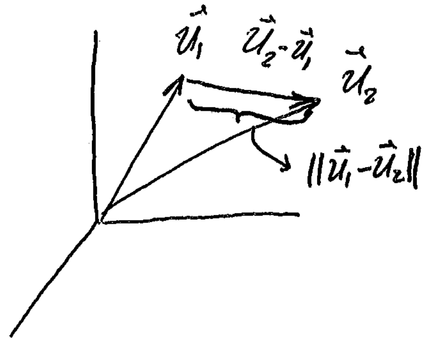
distance between $\vec{u}_1$ and $\vec{u}_2$ = $dist(\vec{u}_1, \vec{u}_2)$

is given by

$$dist(\vec{u}_1, \vec{u}_2) =$$

$$= dist\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}\right) = \sqrt{(1-2)^2 + (0-0)^2 + (-1-2)^2} =$$

$$= \sqrt{10}$$



or $dist(\vec{u}_1, \vec{u}_2) =$

$$= \|(1-2,\ 0-0,\ -1-2)\| = \|\vec{u}_1 - \vec{u}_2\|$$

Def: $\vec{u}, \vec{v} \in \mathbb{R}^n$

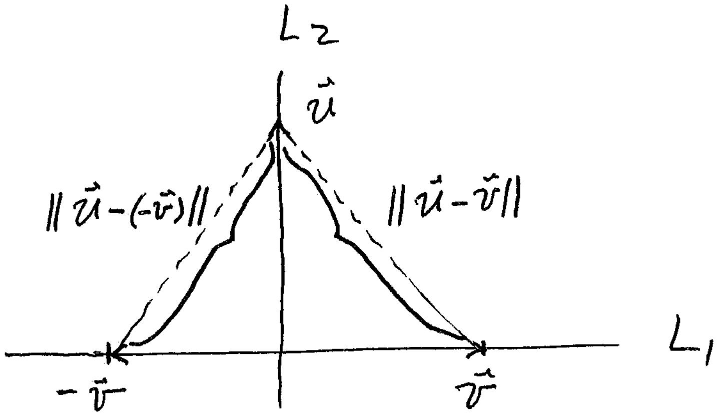
The distance between $\vec{u}$ and $\vec{v}$ is the length of the vector $\vec{u} - \vec{v}$.

$$dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

Notice that if $b \in \mathbb{R}$ and $b = -b \Leftrightarrow b = 0$.

Orthogonal Vectors

From Euclidean geometry, two lines in $\mathbb{R}^2$ or $\mathbb{R}^3$ determined by vectors $\vec{u}$ and $\vec{v}$, respectively



are perpendicular if and only if

$$dist(\vec{u},\vec{v}) = dist(\vec{u},-\vec{v})$$

Now, $\left(dist(\vec{u},\vec{v})\right)^2 = \|\vec{u}-\vec{v}\|^2 =$

$$= (\vec{u}-\vec{v}).(\vec{u}-\vec{v}) = (\vec{u}.\vec{u}) - (\vec{u}.\vec{v}) - (\vec{v}.\vec{u}) + (\vec{v}.\vec{v})$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u}.\vec{v})$$

Similarly,

$$\left(dist(\vec{u},-\vec{v})\right)^2 = \|\vec{u}-(-\vec{v})\|^2 = \|\vec{u}+\vec{v}\|^2$$

$$= (\vec{u}+\vec{v}).(\vec{u}+\vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2(\vec{u}.\vec{v})$$

Therefore, $\left(dist(\vec{u},\vec{v})\right)^2 = \left(dist(\vec{u},-\vec{v})\right)^2 \Leftrightarrow$

$$2(\vec{u}.\vec{v}) = -2(\vec{u}.\vec{v}) \Leftrightarrow \vec{u}.\vec{v} = 0$$

Generalization of this notion to $\mathbb{R}^n$.

<u>Def:</u> <u>$\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal</u> if and only if

$$\boxed{\vec{u} \cdot \vec{v} = 0.}$$

$\vec{0}$ is orthogonal to any $\vec{u} \in \mathbb{R}^n$.

<u>Thm 2</u> (Pythagorean Thm)

Two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal if and only if

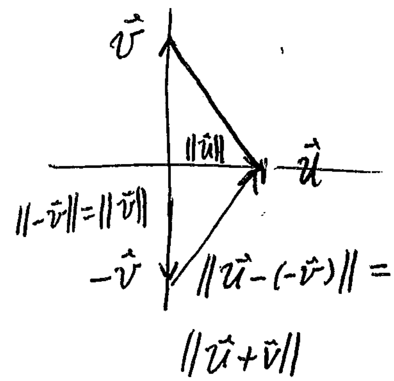
$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

<u>Proof</u>

$$\|\vec{u}+\vec{v}\|^2 \underset{\text{Calculations}}{\overset{\text{from above}}{=\!=\!=}} \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2(\vec{u}\cdot\vec{v})$$

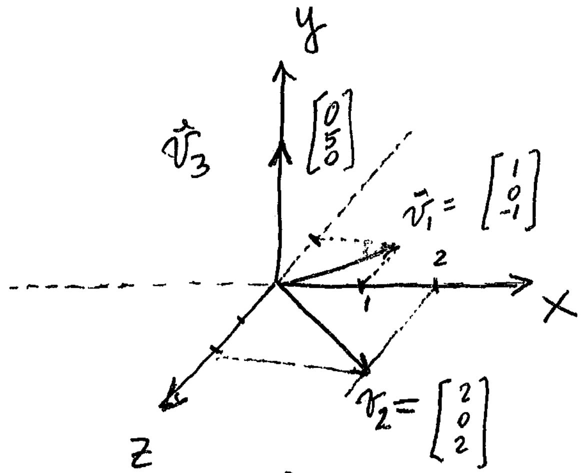
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2$$

if and only if $\vec{u}\cdot\vec{v} = 0$.



Consider the vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$



$$\text{Span}\left\{ \overset{\vec{v}_1}{\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \right\} = \text{ xz plane}$$

and $\vec{v}_3 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$ is such that

$$\vec{v}_3 \cdot \vec{v}_1 = [0\ 5\ 0] \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 0, \quad \vec{v}_3 \cdot \vec{v}_2 = [0\ 5\ 0] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$$

Also, $\vec{v}_3 \cdot (c_1\vec{v}_1 + c_2\vec{v}_2) = c_1(\vec{v}_3 \cdot \vec{v}_1) + c_2(\vec{v}_3 \cdot \vec{v}_2) =$

$$= c_1\, 0 + c_2\, 0 = 0$$

$\Rightarrow$ $\vec{v}_3$ is orthogonal to any vector in xz-plane

we will say $\vec{v}_3$ is orthogonal to W = xz-plane.

Def W Subspace of $\mathbb{R}^n$ and $\vec{z} \in \mathbb{R}^n$

i) If $\vec{z}$ is orthogonal to any vector in W, then $\vec{z}$ is said to be orthogonal to W.

ii) The set of all vectors $\vec{z}$ that are orthogonal to W is called the orthogonal complement of W. It is denoted $W^\perp$.

In our previous example

$$W^\perp = \left\{ \vec{x} \in \mathbb{R}^3 : \ \vec{x} = \alpha \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \right\}.$$

Notice $(W^\perp)^\perp = W$.

Thm. $W \subset \mathbb{R}^n$ is a subspace, $S = \{\vec{v}_1, .., \vec{v}_r\}$ is such that $W = \operatorname{span}(S)$.

i) $\vec{x} \in W^\perp \iff \vec{x}$ is orthogonal to every $\vec{v}_i$ $(i = 1, .., r)$.

ii) $W^\perp$ is a subspace of $\mathbb{R}^n$.

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Find Nul(A) and row(A).

$\vec{x} \in Nul(A)$

$$\Longleftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{array}{l} x_1 = -2x_3 \\ x_2 = -2x_3 \\ x_3 \text{ free} \end{array}$$

$$\Leftrightarrow \vec{x} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

Notice that $\left\{ \begin{array}{l} \vec{r}_1^{\,T} = [1 \; 0 \; 2] \\ \vec{r}_2^{\,T} = [0 \; 1 \; 2] \end{array} \right\}$ satisfies

$$\vec{r}_1 \cdot \vec{x} = \vec{r}_1^{\,T} \vec{x} = [1 0 2] \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$\vec{r}_2 \cdot \vec{x} = \vec{r}_2^{\,T} \vec{x} = [1 0 2] \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \quad \vec{x} \in \left(Row(A)\right)^{\perp}$$

The reverse is also true

if $\vec{x} \in \left(Row(A)\right)^{\perp} \Rightarrow \vec{x} \in Nul(A)$. (Show it!).

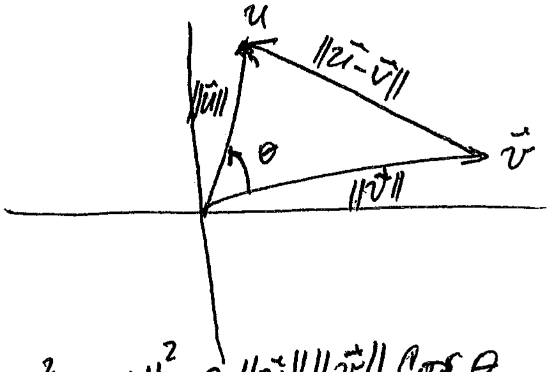
Thm 3. A $m\times n$ matrix then

i) $(\text{Row } A)^{\perp} = \text{Nul}(A)$

ii) $(\text{Col } A)^{\perp} = \text{Nul}(A^T)$.

Proof

Law of Cosines. ($\mathbb{R}^2$ and $\mathbb{R}^3$).



$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\|\vec{u}-\vec{v}\| = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$\Rightarrow \|\vec{u}\|\|\vec{v}\| \cos\theta = \frac{1}{2}\left[\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2\right]$$

$$= \frac{1}{2}\left[u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1-v_1)^2 - (u_2-v_2)^2\right] =$$

$$= \frac{1}{2}\left[2u_1v_1 + 2u_2v_2\right] = u_1v_1 + u_2v_2 = \vec{u}\cdot\vec{v}$$

Therefore,

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}}, \quad \begin{matrix} \vec{u} \neq \vec{0} \\ \vec{v} \neq \vec{0} \end{matrix}$$

This formula is used to define the angle between two vectors in $\mathbb{R}^n$.